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2004 J. Phys. A: Math. Gen. 37 L157

(http://iopscience.iop.org/0305-4470/37/15/L01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) L157–L160 PII: S0305-4470(04)74143-8

LETTER TO THE EDITOR

The adiabatic analogue of the Margolus–Levitin theorem

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Received 7 January 2004
Published 29 March 2004
Online at stacks.iop.org/JPhysA/37/L157 (DOI: 10.1088/0305-4470/37/15/L01)

Abstract
In this letter we derive the minimum time needed for any state of a given quantum system to evolve adiabatically into a distinct (orthogonal) state. This problem is relevant to deriving physical limits in quantum computation and quantum information processing. Our result represents an adiabatic analogue to the Margolus–Levitin theorem (Margolus and Levitin 1998 *Physica* D **120**, 188).

PACS number: 03.67.Lx

In this letter we discuss the question of the minimum time needed for any state of a given quantum system to evolve adiabatically into a distinct (orthogonal) state. The problem of maximum speed of quantum dynamical evolution is relevant to deriving physical limits in quantum computation and quantum information processing. Margolus and Levitin [1] have derived the minimum time required for the non-adiabatic evolution from any state to an orthogonal state of the quantum system. Recently, Lloyd [2] has used the Margolus–Levitin theorem to derive the power/speed limits for the problem of transferring M qubits reliably using a quantum communication channel. It has been shown that by use of entanglement, the M qubits can be transferred $\sqrt{M}$ times more rapidly for the same power as M unentangled qubits. In this letter, we consider the adiabatic case and we derive the Hamiltonian corresponding to the minimum time evolution predicted by the Margolus–Levitin theorem.

Consider a quantum system, which evolves according to the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\xi(t)\rangle = \widehat{H}(t)|\xi(t)\rangle \tag{1}$$

where $\widehat{H}$ is the Hamiltonian of the system (we let $\hbar = 1$). For the non-adiabatic case, we assume that $\widehat{H}$ is time independent. Also, we consider the system in the initial state $|\xi(0)\rangle = |\psi\rangle$, of mean energy $E = \langle\psi\widehat{H}|\psi\rangle$. The Margolus–Levitin theorem [1] asserts that it takes at least a time $T_\perp \geqslant \pi/(2E)$ for the system to evolve from $|\xi(0)\rangle = |\psi\rangle$ to an orthogonal state $|\xi(T_\perp)\rangle = |\varphi\rangle$. This result complements the time–energy uncertainty relation, which requires $T_\perp \geqslant \pi/(2\Delta E)$, where $\Delta E = \sqrt{\langle\psi(\widehat{H} - E)^2|\psi\rangle}$ is the energy spread of the

state. Combining the above two inequalities, one obtains the minimum time required for the evolution to an orthogonal state [1]:

$$T_\perp = T(E, \Delta E) = \max\left(\frac{\pi}{2E}, \frac{\pi}{2\Delta E}\right). \tag{2}$$

Now, let us consider the adiabatic evolution of a quantum system, according to the Schrödinger equation (1). If the Hamiltonian is time independent and the system is initially in its ground state, then it will remain in this state. More specifically, if $|E_0; t\rangle$ and $|E_1; t\rangle$ are, respectively, the ground and first excited states of the Hamiltonian $\widehat{H}(t)$, with energies E_0 and E_1, we define the minimum gap between these eigenvalues

$$\omega_{\min} = \min_{0\leqslant t\leqslant T}[E_1(t) - E_0(t)] \tag{3}$$

and the maximum value of the matrix element of $\mathrm{d}\widehat{H}(t)/\mathrm{d}t$ between the eigenstates as

$$\Omega_{\max} = \max_{0\leqslant t\leqslant T}\left|\left\langle\frac{\mathrm{d}\widehat{H}}{\mathrm{d}t}\right\rangle_{1,0}\right| = \max_{0\leqslant t\leqslant T}\left|\langle E_1; t|\,\frac{\mathrm{d}\widehat{H}}{\mathrm{d}t}|E_0; t\rangle\right|. \tag{4}$$

The adiabatic theorem states that if we prepare the system at time $t = 0$ in its ground state $|E_0; t\rangle$ and let it evolve under the Hamiltonian $\widehat{H}(t)$ for time T, then

$$|\langle E_0; T|\xi(T)\rangle|^2 \geqslant 1 - \varepsilon^2 \tag{5}$$

provided that

$$\Omega_{\max}\omega_{\min}^{-2} \leqslant \varepsilon \tag{6}$$

where $0 < \varepsilon \ll 1$ [3, 4].

This result can be used to design a new type of quantum algorithm based on a time-dependent Hamiltonian [5]. Let us assume that we can build a Hamiltonian for which we know that the ground state encodes the solution of a problem. Then, it suffices to prepare the system in the ground state of another Hamiltonian, easy to build, and progressively change this Hamiltonian to the other one in order to get, after measurement, the sought solution with large probability. The adiabatic theorem imposes the minimum time it takes for this switch to be adiabatic.

The adiabatic method has been studied only for the linear interpolation case, where the 'straight line' interpolation from initial ($\widehat{H}_0$) to final ($\widehat{H}_1$) Hamiltonian is taken [5–7]:

$$\widehat{H}(s) = (1 - s)\widehat{H}_0 + s\widehat{H}_1 \tag{7}$$

where $s = t/T$ is the rescaled time and T is the total time (or *delay schedule*). The initial state of the system $|\xi(0)\rangle = |\psi\rangle$ is the ground state, with energy $E_0 = -E$ ($E > 0$), of the initial Hamiltonian

$$\widehat{H}_0 = -E|\psi\rangle\langle\psi|. \tag{8}$$

We would like to evolve this state, using the linear interpolating Hamiltonian, to the final state $|\xi(1)\rangle = |\varphi\rangle$, which is the ground state, with energy $E_0 = -E$, of the final Hamiltonian

$$\widehat{H}_1(\theta) = -E|\varphi\rangle\langle\varphi|. \tag{9}$$

We assume that $|\psi\rangle$ and $|\varphi\rangle$ are not orthogonal: $\langle\psi|\varphi\rangle = \cos(\theta)$. Therefore we can write

$$|\varphi\rangle = \cos(\theta)|\psi\rangle + \sin(\theta)|\sigma\rangle \tag{10}$$

where $\langle\psi|\sigma\rangle = 0$. In the two-dimensional space spanned by $|\sigma\rangle$ and $|\psi\rangle$, the linear interpolating Hamiltonian reads

$$\widehat{H}(s, \theta) = -E\begin{bmatrix} 1 - s\sin^2(\theta) & \frac{1}{2}s\sin(2\theta) \\ \frac{1}{2}s\sin(2\theta) & s\sin^2(\theta) \end{bmatrix}. \tag{11}$$

The ground state and the first excited state of the above Hamiltonian are

$$E_{0,1}(s,\theta) = \frac{E}{2}\left[1 \mp \sqrt{1 - 4s(1-s)\sin^2(\theta)}\right]. \tag{12}$$

Thus the energy gap is given by

$$\omega(s,\theta) = E\sqrt{1 - 4s(1-s)\sin^2(\theta)}. \tag{13}$$

Also, it is easy to show that the matrix element of $\mathrm{d}\widehat{H}(t)/\mathrm{d}t$ between the eigenstates is given by

$$\Omega(t,\theta) = \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right| \Omega(s,\theta) = \frac{E}{T}\left|\frac{\sin(2\theta)}{2\omega(s,\theta)}\right|. \tag{14}$$

The extreme values are obtained for $s = 1/2$:

$$\omega_{\min}(\theta) = E\cos(\theta) \tag{15}$$

$$\Omega_{\max}(\theta) = \frac{E}{T}|\sin(\theta)|. \tag{16}$$

By substituting (15) and (16) into the adiabaticity condition (6), we obtain the minimum evolution time for the linear interpolation case [5]:

$$T_0(E,\theta) = \frac{1}{E\varepsilon}\frac{|\sin(\theta)|}{\cos^2(\theta)}. \tag{17}$$

A better result can be obtained by assuming that $s = s(t)$ is a smooth function satisfying the boundary conditions: $s(0) = 0, s(T) = 1$ [6, 7]. In this case, the local version of the adiabatic evolution condition reads

$$\left|\frac{\mathrm{d}s}{\mathrm{d}t}\right| \leqslant \varepsilon\frac{\omega^2(s,\theta)}{\Omega(s,\theta)} \tag{18}$$

and the minimum running time can be obtained by integration [6, 7]:

$$T_1(E,\theta) = \int_0^1 \mathrm{d}s\,\frac{\Omega(s,\theta)}{\omega^2(s,\theta)} = \frac{1}{E\varepsilon}\tan(\theta). \tag{19}$$

However, one can see that when the initial and final states are orthogonal, the minimum running time becomes infinite:

$$\lim_{\theta\to\pi/2} T_{0,1}(E,\theta) = \infty. \tag{20}$$

In fact, in the orthogonal case ($\theta = \pi/2$) the minimum gap is $\omega_{\min}(\pi/2) = 0$ and the adiabaticity conditions (6) and (18) are not satisfied. Thus, the adiabatic evolution (based on the linear interpolation method described above) fails when the initial and final states are orthogonal.

In what follows we will show that the adiabatic evolution can be done in finite time by using a simple non-linear interpolation method. There is no reason not to consider non-linear interpolation. The adiabatic algorithm will work taking any path $\widehat{H}(s)$, as long as the adiabaticity condition is satisfied.

We assume that $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal: $\langle\psi|\varphi\rangle = 0$. Now, let us consider the following Hamiltonian:

$$\widehat{H}(s) = -E|\xi(s)\rangle\langle\xi(s)| \tag{21}$$

where

$$|\xi(s)\rangle = \cos\left(\frac{\pi}{2}s\right)|\psi\rangle + \sin\left(\frac{\pi}{2}s\right)|\varphi\rangle \tag{22}$$

is the ground state, with eigenvalue $E_0 = -E(E > 0)$, and $s = t/T \in [0, 1]$.

In the $|\psi\rangle$, $|\varphi\rangle$ basis, the adiabatic Hamiltonian reads

$$\widehat{H}(s) = -E \sin^2\left(\frac{\pi}{2}s\right)|\psi\rangle\langle\psi| - E\cos^2\left(\frac{\pi}{2}s\right)|\varphi\rangle\langle\varphi| - E\sin(\pi s)(|\psi\rangle\langle\varphi| + |\varphi\rangle\langle\psi|). \quad (23)$$

Thus, $|\xi(s)\rangle$ performs an interpolation from the initial wavefunction $|\psi\rangle = |\xi(0)\rangle$ to final wavefunction $|\varphi\rangle = |\xi(1)\rangle$, and $\widehat{H}(s)$ performs an interpolation from the initial Hamiltonian $\widehat{H}_0 = \widehat{H}(0) = -E|\psi\rangle\langle\psi|$ to final Hamiltonian $\widehat{H}_1 = \widehat{H}(1) = -E|\varphi\rangle\langle\varphi|$.

The first excited state is $E_1 = 0$. Thus, the energy gap is constant: $\omega_{\min} = E$. Also, taking into account that

$$\frac{\mathrm{d}\widehat{H}}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\widehat{H}}{\mathrm{d}s} = \frac{1}{T}\frac{\mathrm{d}\widehat{H}}{\mathrm{d}s} \quad (24)$$

the matrix element can be calculated analytically and is given by

$$\Omega_{\max} = \frac{\pi E}{2T}. \quad (25)$$

Thus, the minimum evolution time from $|\psi\rangle$ to $|\varphi\rangle$, in the adiabatic case, is

$$T(E) = \frac{\pi}{2E\varepsilon} \quad (26)$$

where E is the minimum gap between the first excited state and the ground state of the system.

In this letter we have discussed the question of the minimum time needed for any state of a given quantum system to evolve adiabatically into a distinct (orthogonal) state. We have proposed a simple non-linear interpolation method which gives an analogue result to the non-adiabatic Margolus–Levitin theorem.

Acknowledgments

This work was supported by grants from the Defence R&D Canada and Natural Sciences and Engineering Research Council of Canada.

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